

# HIERARCHICAL MULTI-DIMENSIONAL LIMITING STRATEGY ON DG DISCRETIZATION FOR COMPRESSIBLE NAVIER-STOKES EQUATIONS

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## ABSTRACT

The present paper deals with the continuous work of extending multi-dimensional limiting process (MLP) for higher-order discontinuous Galerkin (DG) methods to compute compressible Navier-Stokes equation. From the previous works, it was observed that the MLP shows several superior characteristics, such as efficient controlling of multi-dimensional oscillations and accurate capturing of complex flow structure. Recently, MLP has been extended into DG method for hyperbolic conservation laws. The proposed method, called hierarchical MLP, can be readily extended to convection-dominated problem, such as compressible Navier-Stokes equation. Through several test cases, it is observed that the proposed approach yields outstanding performances in resolving non-compressive as well as compressive viscous flow features.

## INTRODUCTION

Multi-dimensional limiting process (MLP) has been developed quite successfully in finite volume methods (FVM). Compared with traditional limiting strategies, such as TVD or ENO-type schemes, MLP effectively controls unwanted oscillations particularly in multiple dimensions. The theoretical foundation of the MLP limiting strategy is to satisfy the maximum principle to ensure multi-dimensional monotonicity. A series of researches [1] clearly demonstrates that the MLP limiting strategy possesses superior characteristics in terms of accuracy, robustness and efficiency in inviscid and viscous computations on structured and unstructured grids in FVM.

Recently, MLP concept has been successfully extended into discontinuous Galerkin (DG) methods for compressible flows (or hyperbolic conservation laws) on unstructured grids. This approach, called the hierarchical MLP methods, are able to accurately capture complex compressible multi-dimensional flow structure without yielding unwanted oscillations [2]. Particularly, it can provide multi-dimensional monotonic solutions without compromising formal order of accuracy in smooth region. Based on the previous progresses, a robust and accurate but simple form of hierarchical MLP limiting for DG discretization is provided so that it can handle complex vortical structures in compressible viscous flows very accurately, even if such flow features interact with strong shocks.

## MLP AND AUGMENTED MLP CONDITION

The proposed limiting strategy starts from the MLP condition, which is an extension of the one-dimensional monotonic condition. The basic idea of the MLP condition is to control the distribution of both cell-centered and cell-vertex physical properties to mimic the multi-dimensional nature of flow physics.

$$\bar{q}_{v_i}^{\min} \leq q_{v_i} \leq \bar{q}_{v_i}^{\max} \quad (1)$$

where  $q$  is a state variable and  $q_{v_i}$  is the vertex value.  $(\bar{q}_{v_i}^{\min}, \bar{q}_{v_i}^{\max})$  are the minimum and maximum of the cell-averaged values among all neighboring cells sharing the same vertex  $v_i$ . For a linear reconstruction, the MLP-u slope limiters satisfy the maximum principle on the MLP stencil to ensure multi-dimensional monotonicity while maintaining the second order of accuracy on unstructured grids [1].

For higher-order reconstruction, MLP condition is not sufficient to control spurious oscillation. Unlike linear reconstruction, controlling the value on the vertex only does not guarantee the monotonic distribution of the cell. Thus, MLP condition may not identify the troubled-cells which may lead to violation of the maximum principle. After some analysis, a more strict condition, called augmented MLP condition, is proposed to identify maximum-principle violating cell.

$$\bar{q}_{v_i}^{\min} \leq q_{v_i}^{h,\min} \leq q_{v_i} \leq q_{v_i}^{h,\max} \leq \bar{q}_{v_i}^{\max} \quad (2)$$

MLP condition and augmented MLP condition guide to control multi-dimensional oscillations for higher-order reconstruction, which will be discussed in following section.

## HIERARCHICAL MLP LIMITING FOR HIGHER-ORDER DG RECONSTRUCTION

In DG methods, the distribution within a cell is approximated by the sum of shape functions in a suitably smooth function space  $V^n$ , which usually consists of polynomials of order up to  $n$ .

$$\mathbf{q}_j^h(\mathbf{x}, t) = \sum_{i=1}^n \mathbf{q}_j^{(i)}(t) b_j^{(i)}(\mathbf{x}), \quad (3)$$

where  $q_j^h$  is an approximated state variable on the cell  $T_j$  and  $b_j^{(i)}$  is a shape function. The present approach is based on the RKDG method to overcome the stability problem to resolve convective part. To discretize viscous flux accurately and robustly, BR2 [3] method is applied.

In order to handle discontinuous profile, robust limiting algorithm is necessary. Based on the MLP condition and augmented MLP condition, we proposed hierarchical MLP method which consist of two step: At first, the troubled-cells are identified and the augmented MLP condition with extrema detector is used. For these troubled cells, the projection to  $V^{n-1}$  function space or MLP-u slope limiter is applied. This procedure is applied in a hierarchical manner from higher-order mode to  $P2$  mode. Although this approach is originally proposed for hyperbolic conservation laws, it can be applicable to solve compressible Navier-Stokes equation.

## NUMERICAL RESULTS

### Accuracy Test for Two-dimensional Convection-Diffusion Problem

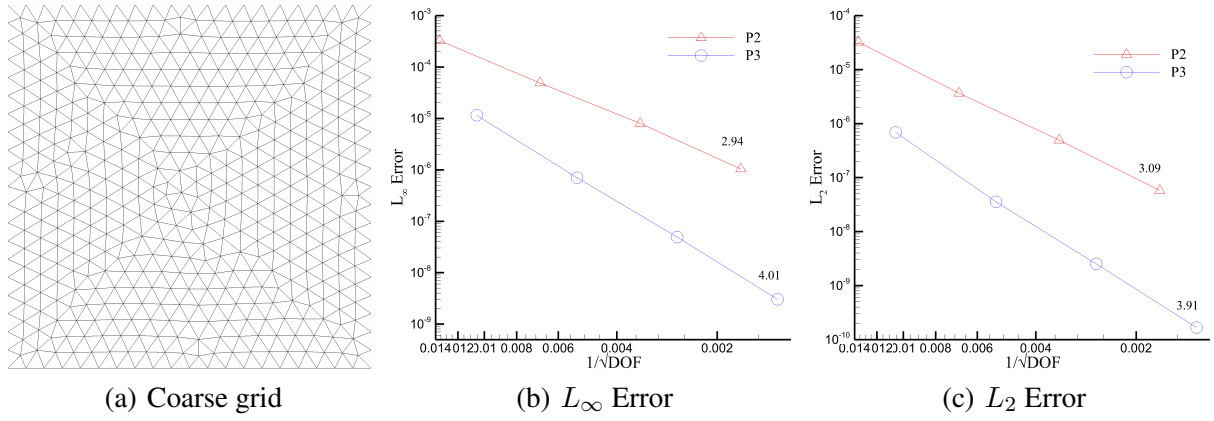


Figure 1. Irregular grid and results of grid refinement test

At first, we examine the grid convergence behavior of the hierarchical MLP scheme by solving two-dimensional convection-diffusion problem.

$$u_t + u_x + u_y = \frac{2a}{\pi^2} u_{xx} + \frac{2a}{\pi^2} u_{yy}, \quad a = 0.02, \quad (4)$$

$$u_0(x, y) = \sin(0.5\pi(x + y)). \quad (5)$$

The analytic solution is  $u(x, y, t) = e^{-at} \sin(0.5\pi(x + y - 2t))$ . Computational domain is  $[-2, 2] \times [-2, 2]$  with periodic boundary condition and irregular grids is used (see Fig. 1(a)). Figs. 1(b) and 1(c) present the computed, which confirms the desired-convergence feature of the proposed method.

### Shock-Mixing Layer Interaction

This problem assess the performance of capturing small scale vortical structure, which interact with shock discontinuity. A spatially developing mixing layer with an initial convective Mach number of 0.6 develops vortices and  $12^\circ$  oblique shock originating from the upper-left corner impinges them. This oblique shock is deflected by the shear layer and then reflects from the bottom slip wall. By interacting downstream vortices and reflected shock, series of shock waves are developed. Detail configuration can be found on Ref. [4]. The computational domain consist of  $[0, 200] \times [-20, 20]$ . Although many filter methods solve this problem with the stretched grid in y direction, the present computation applies the uniformly distributed triangular grids of  $h = 0.75$ . Fig. 2 shows the comparison of pressure contours by FVM, DG method with MLP at  $t = 120$ . While the downstream vortex structures are smeared by FVM, hierarchical MLP on DG discretization capture fine vortical structure without yielding oscillations around shock.

## CONCLUSION

The hierarchical MLP method on DG discretization is successfully extended to compute compressible Navier-Stokes equation. The proposed method is able to accurately capture complex multi-dimensional viscous flow structures without yielding unwanted oscillations. Various numerical results show the desirable characteristics of the proposed limiting strategy, such as multi-dimensional monotonicity, improved accuracy and efficiency.

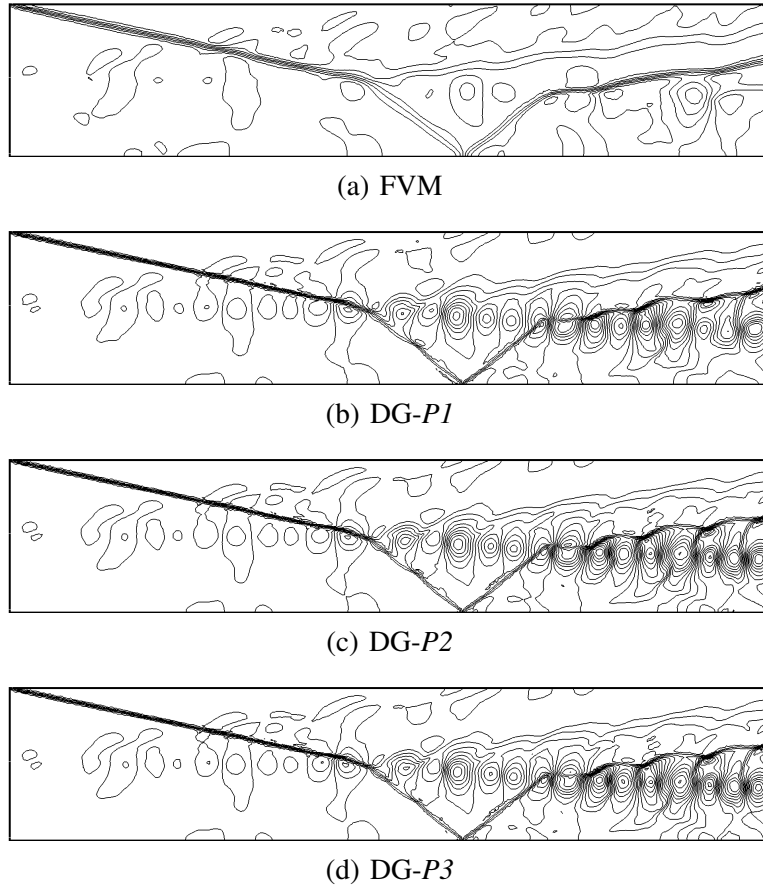


Figure 2. Comparison of pressure contours of shock-mixing layer interaction

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